EXERCICE: ON A COUNTEREXAMPLE RELATIVE TO ABEL'S ANGULAR CONVERGENCE THEOREM

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1. Introduction

Let us recall the following theorem due to Abel (angular convergence theorem):

Theorem 1.1. *Let $S(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power serie of convergence radius R. Let $z_0 = R\, e^{i\theta_0}$ be a complex number such that $|z_0| = R$. We suppose that the serie $S(z_0)$ is convergent. Let $\theta_1 \in [0, \frac{\pi}{2}]$. Let us define the set*

$$A = \{z \in \mathbb{C}\ ;\ |z| < R \text{ and it exists}$$

$$\rho > 0 \text{ and } \theta \in]\theta_0 - \theta_1, \theta_0 + \theta_1[\text{ such that } z = z_0 - \rho\, e^{i\theta}\}.$$

Then $S(z)$ is continuous on $A \cup \{z_0\}$. In particular the following holds

$$\lim_{z \to z_0,\, z \in A} S(z) = S(z_0).$$

We develop now an example showing that the limitation to a subset like A is mandatory. More precisely, if we allow z to approach tangentially z_0 the result is no longer guaranteed.

2. The example

Let $(x_n)_{n>0}$ be the sequence of integers defined by

$$\begin{cases} x_{2p} & = & 3^p \\ x_{2p+1} & = & 2.3^p. \end{cases}$$

We study the power serie

$$S(z) = \sum_{n>0} \frac{z^{x_n}}{n}.$$

1) What is the convergence radius of this power serie?

2) Prove that the serie converges for $z = -1$.

3) Show that we can find a sequence $(z_m)_m$ of complex number such that $|z_m| < 1$, $\lim_{m \to +\infty} z_m = -1$ and $\lim_{m \to +\infty} |S(z_m)| = +\infty$.

Hint: watch the points $z = e^{2i\pi \frac{p}{3^q}}$.

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3. Solution

1) $S(1) = +\infty$ (harmonic serie), then $R \leq 1$. On the other hand, if $|z| < 1$ we have

$$\sum_{n>0} \frac{|z|^{x_n}}{n} \leq \sum_{n>0} |z|^n.$$

Hence the serie converges absolutely for $|z < 1$. We conclude that the convergence radius of the serie is 1.

2) We remark that x_n is odd when n is even and is even when n is odd. Then

$$S(-1) = \sum_{n>0} \frac{(-1)^{x_n}}{n} = \sum_{n>0} \frac{(-1)^{n+1}}{n}$$

which is the alternating harmonic serie. Hence the serie converges for $z = -1$ and moreover $S(-1) = \ln(2)$.

3) Note first that it is possible to find two sequences $(p_m)_{m>0}$ and $(q_m)_{m>0}$ of positive integers such that

$$\lim_{m \to +\infty} \frac{p_m}{3^{q_m}} = \frac{1}{2}.$$

For example let $\left(\frac{r_m}{q_m}\right)_{m>0}$ be the continued fractions of $\log_2(3)$. Then

$$\lim_{m \to +\infty} \frac{r_m}{q_m} = \log_2(3)$$

and moreover

$$\left|\log_2(3) - \frac{r_m}{q_m}\right| < \frac{1}{q_m^2}.$$

We conclude that

$$|q_m \log_2(3) - r_m| < \frac{1}{q_m}.$$

Then

$$\lim_{m \to +\infty} \left(r_m - q_m \log_2(3)\right) = 0,$$

$$\lim_{n \to +\infty} \frac{2^{r_m}}{3^{q_m}} = 1.$$

Let us set $p_m = 2^{r_m - 1}$, we have the desired result:

$$\lim_{m \to +\infty} \frac{p_m}{3^{q_m}} = \frac{1}{2}.$$

Let us set

$$z_m = \lambda_m \, e^{2i\pi \frac{p_m}{3^{q_m}}},$$

where $0 < \lambda_m < 1$. When $n > 2q_m$ we have

$$e^{2i\pi \frac{p_m}{3^{q_m}} x_n} = 1.$$

Hence:

$$S(z_m) = \sum_{n>0} \frac{\lambda_m^{x_n} \, e^{2i\pi \frac{p_m}{3^{q_m}} x_n}}{n} = \sum_{n=1}^{2q_m} \frac{\lambda_m^{x_n} \, e^{2i\pi \frac{p_m}{3^{q_m}} x_n}}{n} + \sum_{n>2q_m} \frac{\lambda_m^{x_n}}{n}$$

and consequently

$$|S(z_m)| \geq \sum_{n>2q_m} \frac{\lambda_m^{x_n}}{n} - 2q_m \geq \sum_{n=2q_m+1}^{K} \frac{\lambda_m^{x_n}}{n} - 2q_m$$

where K is such that

$$\sum_{n=2q_m+1}^{K} \frac{1}{n} > (1+\epsilon)\,(m+2q_m).$$

Let $0 < \lambda_m < 1$, then

$$|S(z_m)| \geq \lambda_m^{x_K} \sum_{n=2q_m+1}^{K} \frac{1}{n} - 2q_m \geq \lambda_m^{x_K}\,(1+\epsilon)\,(m+2q_m) - 2q_m.$$

Let $\eta > 0$ such that $(1-\eta)^{x_K}\,(1+\epsilon) > 1$. If $(1-\eta) < \lambda_m < 1$ we have $|S(z_m)| > m$. Then it is possible to construct a sequence $(\lambda_m)_{m>0}$ such that $0 < \lambda_m < 1$, $\lim_{m\to+\infty} \lambda_m = 1$ and $|S(z_m)| > m$. We can see now that $\lim_{m\to+\infty} z_m = -1$ and $\lim_{m\to+\infty} |S(z_m)| = +\infty$.

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